Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics and Mathematical Finance

Probability Theory and Stochastic Processes – 1st Semester - 2025/2026

Regular Assessment - 16th of December 2025

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(6)* In $\Omega = \{i, s, e, g\}$, consider the set $I = \{\{i\}, \{s\}, \{e\}, \{g\}\} \subset \mathcal{P}(\Omega)$. If $\mathcal{A}(I)$ is the smallest **algebra** containing I, then

$$\#\mathcal{A}(I) = ...$$

(b) *(6)* Consider the set $\mathbb{R}$ endowed with the σ–algebra of the Borelians.

The set $\{5\}$ is a **Borel set** because it can be seen as a countable intersection of borelian sets. Indeed,

$$\{5\} = \bigcap_{n=1}^{\infty}]................, ...]$$

(c) *(8)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians and μ is the **counting** measure. For each $n \in \mathbb{N}$, consider the sets

$$A_n = \{n, n+1, n+2, ...\} \quad \text{and} \quad A = \bigcap_{n=1}^{+\infty} A_n.$$

For this specific case, we have

$$.......... = \mu(A) \neq \lim_{n \to +\infty} \mu(A_n) = ...$$

The inequality does not contradict the **continuity property** because ....

(d) *(8)* Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$. The letters m, δ_a and μ denote the **Lebesgue**, the **Dirac** centered at $a \in \mathbb{R}$ and the **counting** measure, respectively. With respect to the sets

$$A = \left[\frac{1}{4}, 2\right] \quad \text{and} \quad B = [0,1] \cap \mathbb{Q}^c$$

we may say that:

1. $m(B) =$ ....

2. $\sum_{n=1}^{\infty} \delta_{\frac{1}{n}}(A) =$ ....

3. $\mu(B \cap C) = 1$. Then, one possibility for C is ....

4. The set $A \cap \mathcal{V}$ is **non-measurable**.
   Then, one possibility for $\mathcal{V}$ is the .... set.

(e) *(4)* With respect to the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 4$, the **graphical representation** of f^- is:

(f) *(14)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$ and m is the Lebesgue measure. Let $f : [0,1] \to \mathbb{R}$ be:

$$f(x) = \begin{cases} 3x & \text{if } x < 1/2 \\ \\ 3 - 3x & \text{if } x \geq 1/2 \end{cases} .$$

For each $n \in \mathbb{N}$, consider the set ($f^m = f \circ \cdots \circ f$ refers to the composition of maps)

$$\Lambda_n = \{x \in [0,1] : f^m(x) \in [0,1], \forall m \in \{1, ..., n\}\}.$$

Then:

1. the fixed points of f are .... and .... (elements $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$)
2. $\Lambda_2 =$ ....
3. the set Λ_n is the union of .... closed subintervals of $[0,1]$.
4. $m(\Lambda_n) =$..................
5. the set $\Lambda = \bigcap_{n \in \mathbb{N}} \Lambda_n$ is usually called by ....set and $m(\Lambda)=$ ....
6. With respect to the **cardinality** of Λ, it is ....

(g) *(4)* Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians of $\mathbb{R}$ and m is the Lebesgue measure. For $a, b \in \mathbb{R}$ with $a < b$, the **induced measure** induced by the **linear** map $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$m \circ f^{-1}(]a, b]) = \frac{b - a}{5}$$

The analytical expression of f is $f(x) =$ ....

(h) *(4)* Consider the measure space $(\Omega, \mathcal{F}, \mu)$ where Ω is a finite set, $\mathcal{F}$ is an algebra and μ is the **counting** measure. If $A = \{a_1, a_2\} \in \mathcal{F}$ and $f : \Omega \to \mathbb{R}$ is an integrable map, then:

$$\int_A f \, \mathrm{d}\mu = \ldots\ldots\ldots\ldots\ldots$$

(i) *(6)* For each $n \in \mathbb{N}$, define the sequence of simple maps $\varphi_n \equiv \chi_{[n,n+1]} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. In this case, we have:

$$\ldots\ldots\ldots\ldots = \int_{[1,+\infty[} \lim_{n\to+\infty} \varphi_n \, \mathrm{dm} \neq \lim_{n\to+\infty} \int_{[1,+\infty[} \varphi_n \, \mathrm{dm} = \ldots\ldots\ldots$$

This does not contradict the **Monotone Convergence Theorem** because ....

....

(j) *(10)* Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m \times m)$ where $\mathcal{B}(\mathbb{R}^2)$ denotes the $\sigma-$algebra of the Borelians of $\mathbb{R}^2$ and m is the Lebesgue measure on $\mathbb{R}$. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{(x^2 + y^2)^2} & \text{if } 0 < x^2 + y^2 \leq 1 \\ \\ 0 & \text{otherwise} \end{cases} .$$

Using the change of coordinates $x = r\cos\theta$ and $y = r\sin\theta$, $r \in]0, 1]$ and $\theta \in [0, 2\pi]$, we may conclude that:

1. if $r \in]0, 1]$, then $f(r, \theta) =$................................ .

2. $\displaystyle\int_0^{2\pi} \int_0^1 |f(r, \theta)| \times J(r, \theta) \, \mathrm{d}r \, \mathrm{d}\theta =$.. (write explicitly the integrals), which diverges[1].

3. We **cannot** apply the ....Theorem to f because f is not Lebesgue integrable.

[1] $J(r, \theta)$ denotes the Jacobian of the change of variables $(r, \theta) \mapsto (x, y)$.

(k) *(10)* Consider the probability space $\Omega = [0,1]$ endowed with the σ–algebra of borelians $\mathcal{B}([0,1])$. Define the measure

$$\mu = \frac{2}{3}\delta_0 + \frac{1}{3}m,$$

where δ_0 is the **Dirac** measure centered at 0 and m is the **Lebesgue** measure in $[0,1]$. Then:

1. $\mu([0,\frac{1}{3}])$=.........

2. The measures δ_0 and m are .... with respect to μ.

3. ....-.................. theorem says that there exists a map f (integrable) such that

$$\delta_0(A) = \int_A f \,\mathrm{d}\mu$$

4. Using the notation of the previous item, we have $f(0) = \dfrac{d\delta_0}{d\mu}(0) =$ ....

(l) *(9)* Consider the mensurable space $(\mathbb{R}, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra, and

$$X : \mathbb{R} \to \mathbb{N} \cup \{0\}$$

is a random variable. Let F be the **discrete** distribution function associated to $P \circ X^{-1}$ such that for $k \in \mathbb{N} \cup \{0\}$, we have:

$$P(\{\omega \in \Omega : X(\omega) = k\}) = \frac{5^k}{k! e^5}.$$

Let $D = \{a_k\}_{k \in \mathbb{N} \cup \{0\}}$ be the set of **discontinuities** of F ordered by the relation $<$. Then:

1. $\displaystyle\sum_{n=1}^{+\infty} P \circ X^{-1}(\{a_n\}) =$ ....

2. The **characteristic function** associated to X is given by

$$\Phi_X(t) = \text{................................}$$

3. If $E(X) = 5$ and $Var(X) = 5$, then the **Taylor expansion** of Φ_X of degree 2 at $t = 0$ is:

$$P(t) = \text{...........} + \text{............}\, t + \text{............}\, t^2.$$

(m) *(6)* The **Devil's staircase** continuous map is an example of a distribution map $F : \mathbb{R} \to \mathbb{R}$ such that F is not... with respect to the Lebesgue measure. The set of points $x \in [0,1]$ for which $F'(x) = 0$ has Lebesgue measure equal to ....

(n) *(6)* Consider the mensurable space $([0,1], \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra and P is a probability measure. Let $X : [0,1] \to [0,1]$ be a random variable and $\mathcal{G} \subset \mathcal{F}$. Then:

1. the σ–algebra **generated by** X is:

$$\sigma(X) = \{\ldots\ldots\ldots\ldots\ldots\ldots : B \in \mathcal{F}\}$$

2. we say that $\sigma(X)$ and $\mathcal{G}$ are .... if and only if

$$P(A \cap B) = P(A)P(B)$$

for all $A \in \sigma(X)$ and $B \in \mathcal{G}$. In this case, $E(X|\mathcal{G}) =$ ....

(o) *(9)* Consider the following homogeneous Markov chain defined on the state space $\{1, 2\}$ with transition probability matrix:

$$\mathbf{T} = \begin{bmatrix} 1/3 & 2/3 \\ 1/5 & 4/5 \end{bmatrix}$$

Then:

1. Both states of the chain may be classified as...................
(use one of the terms: *"transient"* / *"recurrent null"* / *"recurrent positive"*)

2. The **stationary distribution** associated to T is (.........,.........)

3. The **mean recurrence time** associated to 1 and 2 are equal to .... and ...., respectively.

Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ where $\mathcal{B}(\mathbb{R})$ denotes the σ–algebra of the Borelians and μ is a measure. Suppose that $f : \mathbb{R} \to \mathbb{R}_0^+$ is an integrable map and let $\lambda \in \mathbb{R}^+$. Define the set $A = \{x \in \mathbb{R} : f(x) \geq \lambda\}$. Prove that

$$\mu(A) \leq \frac{1}{\lambda} \int f(x) \, \mathrm{d}\mu(x).$$

2. Consider $(f_n)_{n\in\mathbb{N}}$ the sequence of continuous maps

$$f_n(x) = ne^{-x^3} \sin\left(\frac{x^2}{n}\right), \quad x \in \mathbb{R}_0^+$$

   (a) Identify the map $f : \mathbb{R}_0^+ \to \mathbb{R}$ such that $\lim\limits_{n\to+\infty} f_n(x) = f(x)$.

   (b) Show that $|f_n(x)| \leq x^2 e^{-x^3}$, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$
   (**Remark:** you can use, without proof, that $|\sin(x)| \leq |x|$, for all $x \in \mathbb{R}$)

   (c) Compute $\lim\limits_{n\to+\infty} \int_{[0,+\infty[} f_n(x) \, \mathrm{d}m(x)$, where m is the usual Lebesgue measure.

3. Let $X : \Omega \to \mathbb{R}$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with distribution function

$$F(X) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{4} & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

   If $Z = \sqrt{X}$, compute the **distribution function** of Z.

4. Let $([0,1[, \mathcal{B}([0,1[), m)$ be a space of probability, where m is the Lebesgue measure in the interval $[0,1[$, $X, Y : [0,1[\to \mathbb{R}$ are random variables given by $X(\omega) = 2\omega^2$ and

$$Y(\omega) = \begin{cases} 2\omega & \text{if } 0 \leq \omega < 1/2 \\ 2\omega - 1 & \text{if } 1/2 \leq \omega < 1 \end{cases}.$$

Describe the sets of $\sigma(Y)$ and compute $E(X|Y)$.

5. Given a sequence $(X_n)_n$ of independent and identically distributed random variables with **uniform distribution** on $]0,1]$, compute

$$\lim_{n \to +\infty} \sqrt[n]{X_1 \cdots X_n}.$$

Hint: if necessary, use the change of variables $Y_n = \ln(X_n)$.

DO NOT DO THIS! :)

Credits:

I	II.1	II.2(a)	II.2(b)	II.2(c)	II.3	II.4	II.5
110	15	10	5	15	15	15	15